Understanding quantified statements in mathematics learning at the university

Denik Agustito ¹, Dafid Slamet Setiana ², Heru Sukoco ², Krida Singgih Kuncoro ¹

¹ Universitas Sarjanawiyata Tamaniswai, Yogyakarta, Indonesia
² Universitas Negeri Yogyakarta, Yogyakarta, Indonesia
* Corresponding Author. Email: ragustitoriadi@gmail.com

Abstract: The objective is to provide knowledge for pure mathematics and mathematics education students to understand quantified statements so that they can easily comprehend the meaning of mathematical propositions and facilitate their proof. The method used in this research is a literature review by collecting various sources such as books or scientific writings related to the logic used to understand mathematics, specifically related to quantifiers, both universal and existential quantifiers.

When mathematics and mathematics education students understand the quantified statement \( \forall x [P(x)] \), what they do is take any (cannot choose specific) \( x \), and it must be proven that \( x \) satisfies property \( P \). Then, to understand the quantified statement \( \exists x [P(x)] \), what is done is to choose (at least one) \( x \), and it must be proven that \( x \) satisfies property \( P \). When mathematics and mathematics education students understand the multi-quantifier statement \( \forall x \exists y [P(x, y)] \), what is done is to take any (cannot choose specific) \( x \), then find (at least one) \( y \) (depending on \( x \)), and \( x, y \) must satisfy property \( P \). Then, to understand the multi-quantifier statement \( \exists y \forall x [P(x, y)] \) or \( \exists y [P(x, y) \forall x] \), what is done is to choose \( y \) (independent of \( x \)) and it must be proven that \( y \) satisfies property \( P \) along with all \( x \).

Keywords: Existential; Quantifier; Universal


INTRODUCTION

In the realm of mathematics education at the university level, disciplines such as real analysis, abstract algebra, set theory, and the like necessitate a foundation in logic for the execution of mathematical activities. Logic, as the fundamental underpinning in mathematics, is a program of epistemic nature within a philosophical perspective known as Logicism. Pioneered by prominent figures in 19th-century mathematical logic such as Gottlob Frege and Bertrand Russell, Logicism aims to reduce mathematics to logic. Further exploration of the history and ideas of Gottlob Frege and Bertrand Russell regarding Logicism can be found in (Demopoulos, 2013). In practice, logic, as a discipline, deals with the principles of valid reasoning and arguments. To prove the validity of statements in mathematics, one must employ rules of inference. These rules include modes ponens, modus tollens, hypothetical syllogism, and others. Utilizing these rules, a formal argument is constructed as evidence of validity. Studies on how to test the validity of arguments through rules of inference are typically referred to as logic. For a more in-depth discussion on arguments, their validity, and other logical concepts, refer to (Hurley, 2015; Copi, 1973; Smith, 2003).

For instance, in the study of real analysis, fundamental mathematical concepts like the Archimedean theorem can be expressed as a quantified statement \( \forall x \in \mathbb{R}, \exists n_x \in \mathbb{N}, x < n_x \)
In abstract algebra, basic mathematical concepts such as normal subgroups can be stated as a quantified statement $\forall a \in G, aNa^{-1} \subseteq N$, where $N$ is a normal subgroup of the group $G$ (Herstein, 1986). Similarly, in set theory, fundamental concepts like the regularity axiom in the cumulative hierarchy can be expressed as a quantified statement $\forall x, \exists \alpha, x \in V_\alpha$ (Jech, 2003). If students fail to grasp the concept of quantified statements, their forms, and how to prove mathematical propositions expressed in this form, it becomes an obstacle for them to comprehend mathematics.

To the best of the author's knowledge, several studies have been conducted on the understanding of quantified statements in mathematics education. For instance, Piatek-Jimenez (2010) observed difficulties for undergraduate students in interpreting quantified statements in mathematical propositions. Additionally, Bravo (2023) conducted a dissertation examining students’ understanding of quantifiers and assessing different methods of introducing statements involving quantifiers in a university-level logic context. Based on the author’s teaching experience in introductory courses of real analysis, abstract algebra, and set theory, many students struggle to grasp quantified statements. Drawing on existing research and the author’s teaching experience, this paper aims to provide a guide on understanding quantified statements for students in pure mathematics and mathematics education. The research question addressed in this paper is: How can students in pure mathematics and mathematics education understand quantified statements? This study aims to assist students in comprehending quantified statements, thereby enhancing their ability to understand and prove mathematical propositions.

**METHOD**

The method employed in this research is a literature review involving the collection of various sources, such as books or scholarly writings related to logic used in understanding mathematics, particularly regarding quantifiers, both universal and existential. Key references for this study include (Copi, 1973; Hurley, 2015; Mendelshon, 2005; Miller, 2018; Smith, 2003; Bravo, 2023; Piatek-Jimene, 2010). Subsequently, a critical examination is conducted to elucidate the meaning of quantified statements. The study presents various quantified statements ranging from those found in everyday life to statements encountered in mathematical contexts such as real analysis, abstract algebra, and set theory.

**DISCUSSION**

In this section, we will examine the philosophical foundation of both universal and existential quantifiers so that mathematics and mathematics education students can understand why they need to study them. Historically, the concepts of both universal and existential quantifiers were critically explored for the first time by Gottlob Frege (1848-1925), marking the inception of modern logic (Hersh, 1997). The philosophical foundation begins with an understanding of what logic is. Logic is the study of arguments, and an argument is deemed valid if the truth of its premises guarantees the truth of its conclusion. This means it is impossible for all premises to be true yet lead to a false conclusion. Conversely, an argument is deemed invalid if the truth of its premises does not guarantee the truth of its conclusion. This implies that all premises are true, yet the conclusion is false (Miller, 2018).

For mathematics and mathematics education students, it is essential to grasp the concept of an argument. An argument is a series of supporting statements, consisting of premises and a supported statement, called the conclusion. An argument can be illustrated as follows:

```
p_1
p_2
\vdots
p_n
\hline
\therefore q
```
where \( p_1, p_2, \ldots, p_n \) are the premises, and \( q \) is the conclusion. An example of an argument is as follows:

**Example 1.1.**

1. *Socrates is a human.*
2. *All humans are immortal.*

\[ \therefore \text{Socrates is immortal.} \]

(*) Examine premise (1) and conclusion (3) in **Example 1.1.**

In Frege's logical theory, "Socrates" is considered a *proper name*. Philosophically, what does a proper name mean according to Frege? According to ([Mendelson, 2005](#)), a proper name (Eigenname), or usually referred to as *Eigenname*, is a sign for an object. So, the word *Socrates* refers to a sign for an object. Typically, proper names in Frege's logical theory are symbolized with lowercase letters like \( a, b, c, \ldots, x, y, z \). Since the word "Socrates" is a proper name, it will be symbolized as \( s \). Then, the words "human" and "immortal" in premises (1) and conclusion (3) respectively refer to predicates, and predicates are usually denoted by capital letters like \( A, B, C, \ldots, X, Y, Z \). The words "human" and "immortal" as predicates in order will be symbolized as \( M \) and \( A \). In premises (1) and conclusion (3), they can be symbolized as \( Ms \), meaning Socrates is a human, and \( As \), meaning Socrates is immortal.

(**) Examine premise (2) in **Example 1.1.**

In Frege's logical theory, the word "all" will be symbolized with \( \forall \). Thus, when symbolizing the statement "all humans," the symbol becomes \( \forall x \), where the word "human" in premise (2) is no longer a predicate as in premise (1) but rather a variable symbolized by \( x \). The terminology for variables will generally be understood the same. In Frege's logical theory, the symbol \( \forall x \) is usually referred to as the universal quantifier. Now, how do we symbolize premise (2)? Premise (2) states that all humans are immortal, and this has the same meaning as saying if \( x \) is human, then \( x \) is immortal. So symbolically, premise (2) can be denoted as \( \forall x [Mx \Rightarrow Ax] \), where the symbol \( \Rightarrow \) indicates implication.

(***) Thus, the symbolization of the argument in **Example 1.1** is as follows:

\[
\begin{align*}
Ms \\
\forall x [Mx \Rightarrow Ax] \\
\therefore As
\end{align*}
\]

It has been discussed above that the idea of universal quantifiers arises from an argument. The next step is to explore how to verify the truth of quantifier forms, especially the universal quantifier in the form of \( \forall x [Px]? \).

(i). Students must be able to understand the meaning of a quantified statement \( \forall x [Px] \).

Let's consider a universe of discourse consisting of all living beings in the world. Now, let's examine a statement: "All men in this world are human." With the knowledge discussed earlier, students should be able to symbolize this statement into symbols, resulting in \( \forall x [Lx \Rightarrow Mx] \), where \( L \) represents men, \( M \) represents humans, and \( x \) represents a variable within the universe of discourse, universally quantified and applied to both \( L \) and \( M \). How do we understand \( \forall x [Lx \Rightarrow Mx] \)? This statement can be understood as follows: for every living being, if the living being is a man, then certainly, that living being is human. Is this statement true? Mathematics and mathematics education students should be able to reason. If they encounter anyone who is a man,
they should be able to prove that the man they encounter is indeed human. If students can demonstrate that every man they encounter is human, then the statement “all living beings, if a living being is a man, then certainly, that living being is human” is true; in other words, \( \forall x [Lx \Rightarrow Mx] \) is a true statement. In general, a quantified statement \( \forall x [Px] \) is understood as every/any/all \( x \) must have the property \( P \).

(ii). How can students verify the truth of the quantified statement \( \forall x [Px] \)?

When wanting to verify the truth of the quantified statement \( \forall x [Px] \), mathematics and mathematics education students must demonstrate that each or any or all (not selecting only some) instances of \( x \) satisfy the property \( P \). The form of the quantified statement \( \forall x [Px] \) frequently appears in mathematics, and here are some examples.

**Example 1.2.** Consider the following quantified statement: \( \forall a \in \mathbb{R} [a^2 \geq 0] \). The first step for mathematics and mathematics education students is to understand this quantified statement. As discussed earlier, certainly, the quantified statement can be accepted in reasoning that *for all real numbers, their squares are always non-negative.* The second step is how to verify the truth of this quantified statement. Certainly, what mathematics and mathematics education students do is to take any real number \( x \) (interpreted as all and not selecting only some from real numbers), then prove that \( x^2 \geq 0 \). If mathematics and mathematics education students can prove that for all real numbers \( x \), it satisfies the property \( x^2 \geq 0 \), then the quantified statement \( \forall a \in \mathbb{R} [a^2 \geq 0] \) is true.

Now, consider the following argument:

**Example 1.3.**

(4). *There is something that has both the properties of being red and being square.*

(5). \( \therefore \) *There is something that has the property of being red.*

(*). Consider premises (4) and conclusion (5) in **Example 1.3.**

In Frege's logical theory, the word "there is" will be symbolized with \( \exists \). Thus, when symbolizing the statement "there is something," the symbol becomes \( \exists x \), where the word "something" is a variable symbolized by \( x \). In Frege's logical theory, the symbol \( \exists x \) is usually referred to as the existential quantifier. Next, how do we symbolize premises (4) and conclusion (5)? Premise (4) states that there is something that has both the properties of being red and being square, and this has the same meaning as saying there exists something \( x \) that has the property of being red and the property of being square. So symbolically, premise (4) can be denoted as \( \exists x [Mx \land Px] \), where \( Mx \) represents something red, \( Px \) represents something square, and the symbol \( \land \) is the conjunction connector read as "and." This is similar to conclusion (5), which is symbolized as \( \exists x [Mx] \).

(**). Thus, the symbolization of the argument in **Example 1.3** is as follows:

\[
\exists x [Mx \land Px] \\
\therefore \exists x [Mx]
\]

It has been discussed above that the idea of existential quantifiers arises from an argument. The next step is to explore how to verify the truth of quantifier forms, especially the existential quantifier in the form of \( \exists x [Px] \)?

(i). Students must be able to understand the meaning of a quantified statement \( \exists x [Px] \).

Let's consider a universe of discourse consisting of all living beings in the world. Now, let's examine a statement: "There is a man in this world who is not married." With the
knowledge discussed earlier, students should be able to symbolize this statement into symbols, resulting in $\exists x [Lx \land Tx]$ where $L$ represents men, $T$ represents not married, and $x$ represents a variable within the universe of discourse, existentially quantified and applied to both $L$ and $T$. How do we understand $\exists x [Lx \land Tx]$? This statement can be understood as follows: there exists a living being that is a man, but he is not married. Is this statement true? Mathematics and mathematics education students should be able to reason that they need to find a man (at least one man), but the man they find is not married. If students can prove that there is a man (at least one man) who is not married, then the statement "there is a man in this world who is not married" is true; in other words, $\exists x [Lx \land Tx]$ is a true statement. In general, a quantified statement $\exists x [Px]$ is understood as there exists/some $x$ must have the property $P$.

(ii). How can students verify the truth of the quantified statement $\exists x [Px]$?

When wanting to verify the truth of the existential quantifier statement $\exists x [Px]$, mathematics and mathematics education students must demonstrate that there exists $x$ (at least one $x$) that satisfies the property $P$. The form of the existential quantifier statement $\exists x [Px]$ frequently appears in mathematics, and here are some examples.

**Example 1.4.** Consider the following quantified statement: $\exists a \in \mathbb{R} [a^2 = 1]$. The first step for mathematics and mathematics education students is to understand this quantified statement. As discussed earlier, certainly, the quantified statement can be accepted in reasoning that there exists a real number whose square is equal to one. The second step is how to verify the truth of this quantified statement. Certainly, what mathematics and mathematics education students do is to find or choose a real number $x$, then prove that $x^2 = 1$. If mathematics and mathematics education students can prove that there exists a real number $x$ such that $x^2 = 1$, then the quantified statement $\exists a \in \mathbb{R} [a^2 = 1]$ is true.

The next discussion is on understanding multi-quantifier statements in the form of $\forall x \exists y [P(x, y)]$ and $\exists y \forall x [P(x, y)]$. Now, let’s consider the multi-quantifier statement in the form of $\forall x \exists y [P(x, y)]$ as follows:

**Example 1.5.** Every man must find a woman to marry.

(*) In Example 1.5 above, certainly, if mathematics or mathematics education students want to understand this statement, based on the rules in Frege’s logic, they must examine every man in this world, stating that every man they encounter must have a partner, namely a woman whom he will marry. And it’s clear that the existence of the woman depends on the existence of the man. Can the same woman be a partner for different men? Specifically, it’s allowed, but when speaking generally, different men can have different partners as well.
The statement in Example 1.5 can be symbolized as follows: \( \forall x \exists y [N(x, y)] \) where \( x \) represents a man, \( y \) represents a woman, and \( N(x, y) \) represents \( x \) must marry \( y \), and the existence of \( y \) depends on the existence of \( x \).

Next is how to verify the truth of a multi-quantifier statement in the form of \( \forall x \exists y [P(x, y)] \).

(i). Students must be able to understand the meaning of a multi-quantifier statement in the form of \( \forall x \exists y [P(x, y)] \).

The meaning of a multi-quantifier statement in the form of \( \forall x \exists y [P(x, y)] \) is that all \( x \) must find \( y \) (depending on \( x \)), and \( x, y \) must satisfy the property \( P \).

(ii). How can students verify the truth of a multi-quantifier statement in the form of \( \forall x \exists y [P(x, y)] \)?

When verifying the truth of a multi-quantifier statement in the form of \( \forall x \exists y [P(x, y)] \), mathematics or mathematics education students must choose \( x \) arbitrarily (cannot be selectively chosen), and then find \( y \) (dependent on \( x \)), and prove that \( x \) and \( y \) must satisfy the property \( P \).

Multi-quantifier statements in the form of \( \forall x \exists y [P(x, y)] \) frequently appear in mathematics, and here is one example.

**Example 1.6.** Consider the following multi-quantifier statement: \( \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} [x + y = 0] \). To verify the truth of this multi-quantifier statement, the first step for mathematics or mathematics education students is to take any real number \( x \). The second step is to find a real number \( y \) (depending on \( x \)) and then prove that \( x + y = 0 \). In the second step, one can choose \( y = -x \) and then prove that \( x + y = x + (-x) = 0 \). Thus, it is proven that for any real number \( x \) chosen, there always exists a real number \( y = -x \) such that \( x + y = 0 \); in other words, the proposition \( \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} [x + y = 0] \) is true.

Now consider the multi-quantifier statement in the form \( \exists y \forall x [P(x, y)] \) (or \( \exists y [P(x, y)] \forall x \)) as follows:

**Example 1.7.** A woman must marry every man.

(*) In Example 1.7, certainly if mathematics or mathematics education students want to understand that sentence, then based on the rules in Frege's logic, the student must check whether there is a woman in the world who will marry all the men in the world. And this clearly means the existence of the woman is not dependent (free) on the existence of the man. If students can find a woman who will marry every man in the world, then the statement in Example 1.7 is true.

<table>
<thead>
<tr>
<th>Woman</th>
<th>Man</th>
<th>Marriage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Andi ♥</td>
<td>marries Andi</td>
</tr>
<tr>
<td></td>
<td>Budi ♥</td>
<td>marries Budi</td>
</tr>
<tr>
<td></td>
<td>Carlie ♥</td>
<td>marries Carlie</td>
</tr>
<tr>
<td></td>
<td>x (every)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>♥ marries x</td>
</tr>
</tbody>
</table>
The sentence in Example 1.7 can be symbolized as follows: \( \exists y \ [N(x, y) \ \forall x] \) or \( \exists y \ \forall x \ [N(x, y)] \) where \( x \) represents a man, \( y \) represents a woman, and \( N(x, y) \) states that \( x \) must marry \( y \), and the existence of \( y \) is not dependent (free) on the existence of \( x \).

Next is how to verify the truth of sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \).

(i). Students must understand the meaning of sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \).

The meaning of sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \) is that one must find \( y \) (independent of \( x \)) and \( y \) must satisfy property \( P \) along with all \( x \).

(ii). How can students verify the truth of sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \)?

When verifying the truth of sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \), mathematics or mathematics education students must find \( y \) (must be chosen), and then prove that \( y \) must satisfy property \( P \) along with all \( x \).

Sentences with the form \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \) appear frequently in mathematics, and here is one example.

**Example 1.8.** Consider the multi-quantifier sentence: \( \exists y \in \mathbb{R} \ \forall x \in \mathbb{R} \ [x + y = x] \) or \( \exists y \in \mathbb{R} \ [x + y = x, \ \forall x \in \mathbb{R}] \). To verify the truth of this multi-quantifier sentence, the first step for mathematics or mathematics education students is to choose a real number \( y \). The second step is to prove that \( x + y = x \) for all real numbers \( x \). By choosing \( y = 0 \) and then proving that \( x + y = x + 0 = x \). It is established that there exists a real number \( y = 0 \) such that \( +y = x \) for all real numbers \( x \); in other words, the proposition \( \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} \ [x + y = 0] \) is true.

**CONCLUSION**

The conclusion in this writing is that when mathematics and mathematics education students understand quantified statements \( \forall x \ [P(x)] \), the procedure involves selecting any (not choosing only a few) \( x \) and proving that \( x \) must satisfy property \( P \). Then, to understand quantified statements \( \exists x \ [P(x)] \), the process involves choosing (at least one) \( x \) and proving that \( x \) must satisfy property \( P \). When mathematics and mathematics education students understand multi-quantified statements \( \forall x \exists y \ [P(x, y)] \), the process involves selecting any (not choosing only a few) \( x \), then finding (at least one) \( y \) (depending on \( x \)), and \( x \), \( y \) must satisfy property \( P \). Furthermore, to understand multi-quantified statements \( \exists y \ \forall x \ [P(x, y)] \) or \( \exists y \ [P(x, y) \ \forall x] \), the process involves choosing \( y \) (independent of \( x \)) and proving that \( y \) must satisfy property \( P \) along with all \( x \).

**REFERENCES**


